Math 3010 § 1. **Final Exam** Name:______________
Treibergs $\sigma\pi$ May 1, 2018

1. **Multiple choice.** *The choices are given in the columns "Date / Place" and "Contribution." For each mathematician in the table at the bottom, fill in the number of your choice for the place and date, and a letter for your choice of their contribution.*

Date/Place	**Contribution**
1. 1598 – 1647, Bologna	A. Algebraist, analyst, logician. Proposed 23 fundamental problems.
2. 1601 – 1665, Paris	B. Axiomatic treatment of mechanics, classified cubics, binomial series.
3. 1616 – 1703, Oxford	C. Beta, Gamma, exponential and Zeta functions and applications.
4. 1642 – 1727, Cambridge	D. Calculus of variations: brachystochrone and tautochrone.
5. 1646 – 1716, Hanover	E. Developed algebraic methods to deal with infinite processes.
6. 1654 – 1705, Basel	F. Max where $dy = 0$, inflection where $d^2y = 0$, $d(uv) = u\,dv + v\,du$.
7. 1707 – 1783, Berlin	G. Method of indivisibles to compare areas and volumes.
8. 1854 – 1912, Paris	H. Slope of tangent to curve. Probability.
9. 1862 – 1943, Göttingen	I. Studied complex analysis, mathematical physics, DE's, analysis situs.

Mathematician	Date / Place	Contribution
Johann Bernoulli	6	D
Bonaventura Cavalieri	1	G
Leonhard Euler	7	C
Pierre de Fermat	2	H
David Hilbert	9	A
Gottfried Wilhelm Leibnitz	5	F
Isaac Newton	4	B
Henri Poincaré	8	I
John Wallis	3	E

2. **Short Answer.** *Here a list of regions where mathematics developed. For each region, identify an important mathematician of that region, state an important discovery by this mathematican, and state a feature of the mathematics of that region that distinguishes it from other regions.*

Many answers are possible.

Region	Mathematician	Their Discovery	Distinguishing Feature
Greece	Euclid	"Elements" was model of logical development	Dealt with irrationals using geometric arguments
China	Zhū Shijié	Eliminating variables method in simultaneaous nonlinear equations	Computational algorithms using decimal place numbers
India	Bhâskara II	General solution to Pell's equation	Progressed finding integer solutions to indeterminate equations
Islamic Region	al-Khwārizmī	Geometric treatment of quadratic equation	Transmitted forgotten Greek and Indian mathematics to Europe
Europe	François Viète	Solved cubic using circular functions	First advance of mathematics beyond the Greeks

3. *Determine whether the following statements are true or false. Give a detailed explanantion of ONE of your answers (a)–(d).*

   (a) *There are infinitely many rational solutions of the Diophantine equation* $x^2 + y^2 = 1$.

   TRUE. Consider a line through the point $(-1, 0)$ with rational slope t. Solve where it intersects the circle.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y &= t(x+1) \end{aligned}$$

   Thus

$$x^2 + t^2(x+1)^2 = 1$$

   or

$$(t^2+1)x^2 + 2t^2x + (t^2 - 1) = 0.$$

   Since $x = -1$ is a zero, this factors

$$(x+1)((t^2+1)x - (t^2-1)) = 0$$

   so

$$x = \frac{t^2-1}{t^2+1} \quad \Longrightarrow \quad y = t(x+1) = \frac{2t}{t^2+1}$$

   Since there are infinitely many rational t, and since x and y are rational functions of t so are themselves rational, this parameterization gives infinitely many rational solutions of $x^2 + y^2 = 1$.

   (b) *The cubic equation* $y^3 = 3y + 8$ *may be solved by radicals.*

   TRUE. Use Cardano's solution of the cubic: suppose $y = u + v$. Then

$$y^3 = u^3 + v^3 + 3uvy = 3y + 8.$$

   Cardanos trick is to solve separately

$$\begin{aligned} u^3 + v^3 &= 8 \\ 3uv &= 3. \end{aligned}$$

   Put $v = \dfrac{1}{u}$ so that

$$u^3 + \frac{1}{u^3} = 8$$

   which yields the quadratic equation

$$(u^3)^2 - 8u^3 + 1 = 0.$$

   Its solution is

$$u^3 = \frac{8 + \sqrt{64-4}}{2} = 4 + \sqrt{15}; \qquad v^3 = 4 - \sqrt{15}.$$

   Thus the solution of the cubic is given by radicals as

$$y = \sqrt[3]{4+\sqrt{15}} + \sqrt[3]{4-\sqrt{15}}.$$

(c) *The real projective plane $\mathbb{RP}^2$ (the space of all lines through the origin in $\mathbb{R}^3$) is fundamentally different than the two sphere $\mathbb{S}^2$.*

TRUE. A line through the origin is determined by a unit vector $u \in \mathbb{S}^2$. The antipodal point $-u$ determines the same line so $\mathbb{RP}^2$ is the sphere with antipodal points identified. On the sphere, any simple closed curve separates the sphere into two parts. But on $\mathbb{RP}^2$, the equator does not separate the upper hemisphere from the lower, because the hemispheres are the same place under the antipodal identification. A point $P = (x, y, z)$ in the upper hemisphere near the equator has $0 < z$ small and positive is identified to (the same point in $\mathbb{RP}^2$) as $P' = (-x, -y, -z)$ in the lower hemisphere.

(d) *The kth pentagonal number is $p_k = \dfrac{3k^2 - k}{2}$.*

TRUE. $p_1 = \dfrac{3 \cdot 1^2 - 1}{2} = 1$, $p_2 = \dfrac{3 \cdot 2^2 - 2}{2} = 5$ so the first two terms are correct. (see p. 39 of the text.) At each step, the number of points per side is increased. Thus the number added is is $k+1$ more points to each of three sides, with the two corner points in the middle double counted, thus

$$p_{k+1} = p_k + 3(k+1) - 2 = p_k + 3k + 1$$

For example $p_3 = p_2 + 7 = 5 + 7 = 12$ and $p_4 = p_3 + 10 = 12 + 10 = 22$. The proof is by induction. We have already checked the base case $p_1 = 1$. Now assume that the kth number is correct. Then, using the induction hypothesis,

$$\begin{aligned} p_{k+1} &= p_k + 3k + 1 \\ &= \frac{3k^2 - k}{2} + 3k + 1 \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3(k^2 + 2k + 1) - k - 1}{2} \\ &= \frac{3(k+1)^2 - (k+1)}{2} \end{aligned}$$

and the induction step is proved.

4. (a) *A sequence is defined from a starting number a_0, and then by the recursion*

$$a_{k+1} = b a_k + c, \qquad \text{for } k \ge 1,$$

where b and c are constants. Find a closed form for the generating function $f(x)$ for the sequence $a_0, a_1, a_2, \ldots$.

Such a sequence occurs as the amount of money in a bank account which earns periodic interest at the rate $b - 1$ and gets a periodic deposit of c. As we did for the Fibonacci Sequence, the generating function is

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

We have, substituting $j = k + 1$,

$$x f(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{j=1}^{\infty} a_{j-1} x^j.$$

It follows from the recursion formula

$$f(x) - bxf(x) = a_0 + \sum_{j=1}^{\infty}(a_j - ba_{j-1})x^j = a_0 + \sum_{j=1}^{\infty} cx^j$$

from which it follows that

$$(1 - bx)f(x) = a_0 + c\left(\frac{1}{1-x} - 1\right) = a_0 + \frac{cx}{1-x}.$$

We conclude

$$f(x) = \frac{a_0}{1-bx} + \frac{cx}{(1-x)(1-bx)}.$$

(b) *Following Newton, find the power series for* $f(x) = \dfrac{1}{\sqrt[3]{1-x^2}}$.

The binomial series is

$$(1+y)^p = \sum_{k=0}^{\infty}\binom{p}{j}y^j.$$

Here $y = -x^2$ and $p = -\dfrac{1}{3}$. Computing the first four binomial coefficients,

$$\binom{-\frac{1}{3}}{0} = 1; \quad \binom{-\frac{1}{3}}{1} = -\frac{1}{3};$$

$$\binom{-\frac{1}{3}}{2} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!} = \frac{4}{3^2 \cdot 2!}; \quad \binom{-\frac{1}{3}}{3} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!} = -\frac{28}{3^3 \cdot 3!}$$

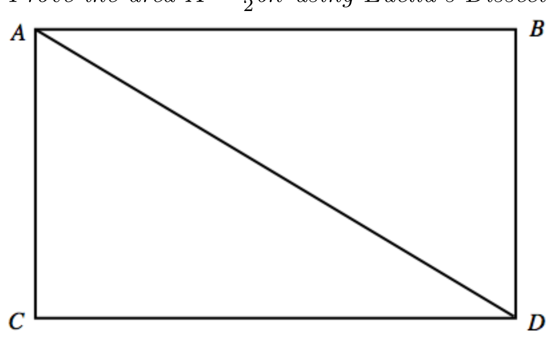
Hence

$$(1+y)^{-\frac{1}{3}} = 1 - \frac{1}{3}y + \frac{4}{3^2 \cdot 2!}y^2 - \frac{28}{3^3 \cdot 3!}y^3 + \cdots$$

$$\frac{1}{\sqrt[3]{1-x^2}} = 1 + \frac{1}{3}x^2 + \frac{4}{3^2 \cdot 2!}x^4 + \frac{28}{3^3 \cdot 3!}x^6 + \cdots$$

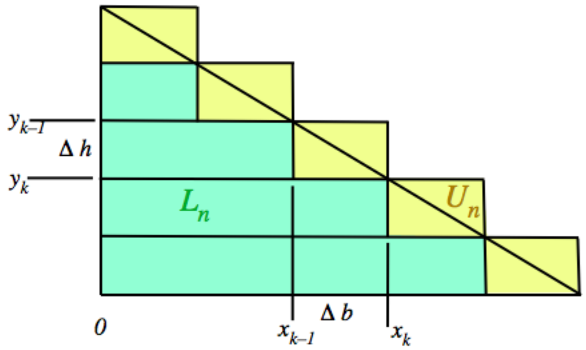
5. *Let T be a triangle whose vertices are $(0,0)$, $(b,0)$ and $(0,h)$ with base b and height h has area $A = \frac{1}{2}bh$.*

(a) *Prove the area $A = \frac{1}{2}bh$ using Euclid's Dissection Method.*



If we make a congruent copy of the right triangle $\triangle ACD$ and glue it to $\triangle DBA$ we get a rectangle $ABCD$ of area bh with two congruent nonoverlapping subtriangles of equal area. Thus $2A(\triangle ACD) = bh$.

(b) *Prove the area $A = \frac{1}{2}bh$ using Eudoxus' Method of Exhaustion.*



The use of Eudoxus Method for triangles is illustrative only because the volume of triangles is known from other means. Let T be the triangle. Let us consider a lower sum and an upper sum as in the sense of Riemann integration. We subdivide the vertical side into n equal strips of height $\Delta h = \dfrac{h}{n}$. For each $k = 1, \ldots, n$ the strip falls between y_{k-1} and y_k where $y_k = \dfrac{kh}{n}$. The inner (green) staircase L_n is the union of strips $[y_{k-1}, y_k] \times [0, x_{k-1}]$ and the outer (yellow) staircase U_n is the union of longer strips $[y_{k-1}, y_k] \times [0, x_k]$, where $x_k = \dfrac{kb}{n}$. Their areas are

$$A(L_n) = \sum_{k=1}^{n} \frac{(k-1)b}{n} \cdot \frac{h}{n} = \frac{hb}{n^2} \sum_{k=1} (k-1) = \frac{hbn(n-1)}{2n^2}$$

$$A(U_n) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{h}{n} = \frac{hb}{n^2} \sum_{k=1}^{n} k = \frac{hbn(n+1)}{2n^2}$$

$$A(U_n) - A(L_n) = \frac{hb}{n}$$

Now do Eudoxus' proportion argument. Suppose there were a number S less than the proposed area $S < \frac{1}{2}bh$. Then the area of triangle is larger than S. To see it, choose n so large that $A(U_n) - A(L_n) < \frac{1}{2}bh - S$. Indeed, since L_n is contained in the triangle,

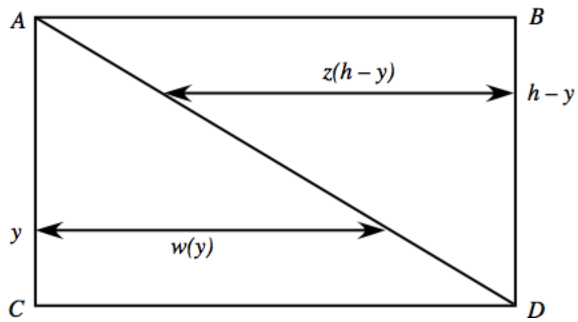
$$S = \frac{1}{2}bh - \left(\frac{1}{2}bh - S\right) < \frac{hbn(n+1)}{2n^2} - (A(U_n) - A(L_n)) = A(L_n) < A(T).$$

Similarly, suppose there were a number W greater than the proposed area $W > \frac{1}{2}bh$. Then the area of triangle is smaller than W. To see it, choose n so large that $A(U_n) - A(L_n) < W - \frac{1}{2}bh$. Since U_n is contains the triangle,

$$W = \frac{1}{2}bh + \left(W - \frac{1}{2}bh\right) > \frac{hbn(n-1)}{2n^2} + (A(U_n) - A(L_n)) = A(U_n) > A(T).$$

We conclude $A(T) = \frac{1}{2}bh$.

(c) *Prove the area* $A = \frac{1}{2}bh$ *using Cavalieri's' Principle.*



Cavalieri's principle says that two figures have the same area if their slices have the same length. Thus if y is the distance to C then the width of the triangle $\triangle ACD$ at height y is

$$w(y) = b - \frac{b}{h}y$$

On the other hand, the width of $\triangle DBA$ at height $h - y$ is

$$z(h-y) = b - \left(b - \frac{b}{h}(h-y)\right) = b - \frac{b}{h}y$$

which is the same. Thus the areas of the two triangles are equal

$$\int_0^h w(y)\,dy = \int_h^0 z(h-y)\,d(h-y).$$

It follows as in (a),

$$2A(\triangle ACD) = A(\triangle ACD) + A(\triangle DBA) = A(ABCD) = bh.$$

6. (a) *Using Fermat's method of ad-equality, find the slope of the tangent line at* $x > 1$ *of the curve*

$$y = x^4 - x.$$

Let the tangent line to the curve at $(x, f(x))$ cross the x-axis at t. Then the slopes of the triangles at two infinitesimally nearby points are equal to the slope

$$\frac{f(x)}{x-t} = \frac{f(x+E)}{x+E-t}.$$

Cross multiplying

$$(x^4 - x)(x + E - t) = \Big((x+E)^4 - (x+E)\Big)(x-t)$$

expanding

$$(x^4 - x)(x + E - t) = \Big(x^4 + 4x^3E + 6x^2E^2 + 4xE^3 + E^4 - x - E\Big)(x-t)$$

cancelling

$$(x^4 - x)E = \Big(4x^3E + 6x^2E^2 + 4xE^3 + E^4 - E\Big)(x-t)$$

and dividing by E yields

$$x^4 - x = (4x^3 - 1)(x-t) + \Big(6x^2E + 4xE^2 + E^3\Big)(x-t).$$

Now, since E is small we drop any remaining E terms to find

$$x^4 - x = (4x^3 - 1)(x - t)$$

so the slope is

$$\frac{x^4 - x}{x - t} = 4x^3 - 1.$$

(b) *Use Newton's method of fluxions or Leibnitz's calculus of differentials to find the slope of the same curve*

$$y = x^4 - x.$$

Newton viewed the equation dynamically. His fluxion is the rate of change of a variable $\dot{x}$ and his infinitesimals which he called moments of fluxion, are $\dot{x}o$, where o is an infinitely small quantity. So both sides change in a constrained fashion

$$y + \dot{y}o = (x + \dot{x}o)^4 - (x + \dot{x}o).$$

Expanding

$$y + \dot{y}o = x^4 + 4x^3\dot{x}o + 6x^2\dot{x}^2o^2 + 4x\dot{x}^3o^3 + \dot{x}^4o^4 - x - \dot{x}o.$$

Since $y = x^4 - x$ we get

$$\dot{y}o = +4x^3\dot{x}o + 6x^2\dot{x}^2o^2 + 4x\dot{x}^3o^3 + \dot{x}^4o^4 - \dot{x}o.$$

Dividing by o yields

$$\dot{y} = +4x^3\dot{x} - \dot{x} + 6x^2\dot{x}^2o + 4x\dot{x}^3o^2 + \dot{x}^4o^3.$$

Now since o is negligible

$$\dot{y} = +4x^3\dot{x} - \dot{x}$$

so the slope is

$$\frac{\dot{y}}{\dot{x}} = 4x^3 - 1.$$

Leibnitz was even more modern.

$$dy = d(x^4 - x) = d(x^4) - dx = 4x^3\,dx - dx$$

so the slope equals

$$\frac{dy}{dx} = 4x^3 - x.$$

7. (a) *Use Euler's method to find* $S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$.

Hint: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\cdots$

By changing variables $y = x^2$ we see that

$$\cos\sqrt{y} = 1 - \frac{y}{2!} + \frac{y^2}{4!} - \frac{y^3}{6!} + \cdots = \left(1 - \frac{y}{\frac{\pi^2}{4}}\right)\left(1 - \frac{y}{\frac{3^2\pi^2}{4}}\right)\left(1 - \frac{y}{\frac{5^2\pi^2}{4}}\right)\cdots$$

whose zeros are

$$y = \frac{\pi^2}{4}, \quad \frac{3^2\pi^2}{4}, \quad \frac{5^2\pi^2}{4}, \quad \ldots$$

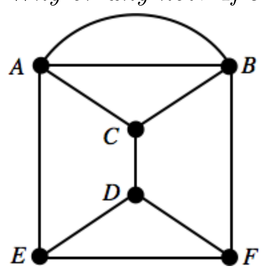
Euler assumed that the negative y^1 coefficient gave the sum of the reciprocals

$$\frac{1}{2!} = \frac{1}{\frac{\pi^2}{4}} + \frac{1}{\frac{3^2\pi^2}{4}} + \frac{1}{\frac{5^2\pi^2}{4}} + \cdots$$

In other words

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(b) *Is there a path in this graph that crosses each edge exactly once (an Eulerian path)? Why or why not? If so, what is such a path?*



THERE IS NO EULERIAN PATH. The graph has four vertices with odd degree, C, D, E and F. A path must start or end at a vertex with odd degree. But a path only has two starting or ending points, thus it cannot start/end at four points.

8. *Using Newton's version of Newton's method, starting from $x_0 = 2$, do at least three iterations of the algorithm to approximate the positive zero of $x^2 - 3$.*

We suppose $x = 2 + p$ and substitute.

$$0 = x^2 - 3 = (p + 2)^2 - 3 = p^2 + 4p + 1.$$

Neglecting the p^2 term we get the first correction equation

$$0 = 4p + 1 \quad \Longrightarrow \quad p = -\frac{1}{4}.$$

Thus $x_0 + p = \boxed{x_1 = 2 - \frac{1}{4} = \frac{7}{4}.}$

We suppose $p = -\frac{1}{4} + q$ and substitute.

$$0 = p^2 + 4p + 1 = \left(-\frac{1}{4} + q\right)^2 + 4\left(-\frac{1}{4} + q\right) + 1 = \frac{1}{16} + \frac{7}{2}q + q^2$$

Neglecting the q^2 term we get the second correction equation

$$0 = \frac{1}{16} + \frac{7}{2}q \quad \Longrightarrow \quad q = -\frac{1}{56}.$$

Thus $x_0 + p + q = \boxed{x_2 = 2 - \frac{1}{4} - \frac{1}{56} = \frac{97}{56}.}$

We suppose $q = -\frac{1}{56} + r$ and substitute.

$$0 = \frac{1}{16} + \frac{7}{2}q + q^2 = \frac{1}{16} + \frac{7}{2}\left(-\frac{1}{56} + r\right) + \left(-\frac{1}{56} + r\right)^2 = \frac{1}{56^2} + \frac{97}{28}r + r^2$$

Neglecting the r^2 term we get the third correction equation

$$0 = \frac{1}{56^2} + \frac{97}{28} r \qquad \Longrightarrow \qquad r = -\frac{1}{2 \cdot 56 \cdot 97} = -\frac{1}{10864}.$$

Thus $x_0 + p + q + r = \boxed{x_3 = 2 - \frac{1}{4} - \frac{1}{56} - \frac{1}{10864} = \frac{18817}{10864}.}$

9. *Find all integers x that simultaneously satisfy the congruences.*

$$\begin{aligned} x &\equiv 1 \quad \text{mod } 5 \\ x &\equiv 3 \quad \text{mod } 6 \\ x &\equiv 5 \quad \text{mod } 7 \end{aligned}$$

To satisfy the first congruence, $x = 1 + 5y$ for some integer y. To satisfy the second,

$$1 + 5y = 3 - 6z$$

for some integer z, hence

$$5y + 6z = 2$$

Since $5(-1) + 6(1) = 1$, all solutions are given by

$$y = -2 + 6t, \qquad z = 2 - 5t$$

for some integer t. This implies

$$x = 1 + 5(-2 + 6t) = -9 + 30t.$$

Finally to satisfy the last congruence,

$$-9 + 30t = 5 - 7u$$

for some integer u. In other words

$$30t + 7u = 14.$$

Because $30 \cdot 1 + 7(-4) = 2$ so $30 \cdot 7 + 7(-28) = 14$, we have a general solution

$$t = 7 + 7v, \qquad u = -28 - 30v$$

for some integer v. It follows that all solutions are of the form

$$x = -9 + 30t = -9 + 30(7 + 7v) = 201 + 210v.$$

We check:

$$201 = 40 \cdot 5 + 1 = 33 \cdot 6 + 3 = 28 \cdot 7 + 5.$$